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Geometric Quantization of Relativistic Hamiltonian Mechanics

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A relativistic Hamiltonian mechanical system is seen as a conservative Dirac constraint system on the cotangent bundle of a pseudo-Riemannian manifold. We provide geometric quantization of this cotangent bundle where the quantum constraint serves as a relativistic quantum equation.

KEY WORDS: geometric quantization; quantum constraint; relativistic mechanics.

Both relativistic and nonrelativistic mechanical systems on a configuration space Q can be seen as conservative Dirac constraint systems on the cotangent bundle T^*Q of Q, but occupy its different subbundles. Therefore, one can follow the precedent of geometric quantization of nonrelativistic time-dependent mechanics in order to quantize relativistic mechanics.

Recall that, given a symplectic manifold (Z, Ω) and a Hamiltonian H on Z, a Dirac constraint system on a closed imbedded submanifold $i_N : N \to Z$ of Z is defined as a Hamiltonian system on N provided with the pullback presymplectic form $\Omega_N = i_N^*\Omega$ and the pullback Hamiltonian i_N^*H (Gotay *et al.*, 1978; Mangiarotti and Sardanashvily, 1998; Muñoz-Lecanda, 1989). Its solution is a vector field γ on N, which fulfils the equation

$$\gamma\rfloor\Omega_N + i_N^*\, dH = 0.$$

Let N be coisotropic. Then a solution exists if the Poisson bracket $\{H, f\}$ vanishes on N whenever f is a function vanishing on N. It is the Hamiltonian vector field of H on Z restricted to N.

A configuration space of nonrelativistic time-dependent mechanics (henceforth NRM) of m degrees of freedom is an $(m + 1)$-dimensional smooth fibre bundle $Q \to \mathbb{R}$ over the time axis $\mathbb{R}$ (Mangiarotti and Sardanashvily, 1998; Sardanashvily, 1998). It is coordinated by $(q^\lambda) = (q^0, q^i)$, where q^0 is the standard

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Cartesian coordinate on $\mathbb{R}$. Let T^*Q be the cotangent bundle of Q equipped with the induced coordinates $(q^\lambda, p_\lambda = \dot{q}_\lambda)$ with respect to the holonomic coframes $\{dq^\lambda\}$. Provided with the canonical symplectic form

$$\Omega = dp_\lambda \wedge dq^\lambda, \tag{1}$$

the cotangent bundle T^*Q plays the role of a homogeneous momentum phase space of NRM. Its momentum phase space is the vertical cotangent bundle V^*Q of $Q \to \mathbb{R}$ coordinated by (q^λ, q^i). A Hamiltonian $\mathcal{H}$ of NRM is defined as a section $p_0 = -\mathcal{H}$ of the fibre bundle $T^*Q \to V^*Q$. Then the momentum phase space of NRM can be identified with the image N of $\mathcal{H}$ in T^*Q, which is the one-codimensional (consequently, coisotropic) imbedded submanifold given by the constraint

$$\mathcal{H}_T = p_o + \mathcal{H}(q^\lambda, p_k) = 0.$$

Furthermore, a solution of a nonrelativistic Hamiltonian system with a Hamiltonian $\mathcal{H}$ is the restriction γ to $N \cong V^*Q$ of the Hamiltonian vector field of $\mathcal{H}_T$ on T^*Q. It obeys the equation $\gamma\rfloor\Omega_N = 0$ (Mangiarotti and Sardanashvily, 1998; Sardanashivily, 1998). Moreover, one can show that geometric quantization of V^*Q is equivalent to geometric quantization of the cotangent bundle T^*Q, where the quantum constraint $\widehat{\mathcal{H}}_T\psi = 0$ on sections ψ of the quantum bundle serves as the Schrödinger equation (Giachetta *et al.*, 2002a,b). This quantization is a variant of quantization of presymplectic manifolds via coisotropic imbeddings (Gotay and Śniatycki, 1981).

A configuration space of relativistic mechanics (henceforth RM) is an oriented pseudo-Riemannian manifold (Q, g), coordinated by (q^λ). Its momentum phase space is the cotangent bundle T^*Q provided with the symplectic form Ω (1). Note that one also considers another symplectic form $\Omega + F$, where F is the strength of an electromagnetic field (Śniatycki, 1980). A relativistic Hamiltonian is defined as a smooth real function H on T^*Q (Mangiarotti and Sardanashvily, 1998; Rovelli, 1991; Sardanashvily, 1998). Then a relativistic Hamiltonian system is described as a Dirac constraint system on the subspace N of T^*Q given by the equation

$$H_T = g_{\mu\nu}\partial^\mu H \partial^\nu H - 1 = 0. \tag{2}$$

Similarly to geometric quantization of NRM, we provide geometric quantization of the cotangent bundle T^*Q and characterize a quantum relativistic Hamiltonian system by the quantum constraint

$$\widehat{\mathcal{H}}_T\psi = 0. \tag{3}$$

We choose the vertical polarization on T^*Q spanned by the tangent vectors ∂^λ. The corresponding quantum algebra $\mathcal{A} \subset C^\infty(T^*Q)$ consists of affine functions of momenta

$$f = a^\lambda(q^\mu)p_\lambda + b(q^\mu) \tag{4}$$

on T^*Q. They are represented by the Schrödinger operators

$$\hat{f} = -ia^\lambda\partial_\lambda - \frac{i}{2}\partial_\lambda a^\lambda - \frac{i}{4}a^\lambda\partial_\lambda \ln(-g) + b, \quad g = \det(g_{\alpha\beta}), \tag{5}$$

in the space $\mathbb{C}^\infty(Q)$ of smooth complex functions on Q.

Note that the function H_T (2) need not belong to the quantum algebra $\mathcal{A}$. Nevertheless, one can show that, if H_T is a polynomial of momenta of degree k, it can be represented as a finite composition

$$H_T = \sum_i f_{1i} \dots f_{ki} \tag{6}$$

of products of affine functions (4), i.e., as an element of the enveloping algebra $\overline{\mathcal{A}}$ of the Lie algebra $\mathcal{A}$ (Giachetta *et al.*, 2002a). Then it is quantized

$$H_T \mapsto \widehat{H}_T = \sum_i \hat{f}_{1i} \dots \hat{f}_{ki} \tag{7}$$

as an element of $\overline{\mathcal{A}}$. However, the representation (6) and, consequently, the quantization (7) fail to be unique.

Let us provide the above mentioned formulation of classical RM as a constraint autonomous mechanics on a pseudo-Riemannian manifold (Q, g) (Giachetta *et al.*, 1999; Mangiarotti and Sardanashvily, 1998, 2000). Note that it need not be a space-time manifold.

The space of relativistic velocities of RM on Q is the tangent bundle TQ of Q equipped with the induced coordinates $(q^\lambda, \dot{q}^\lambda)$ with respect to the holonomic frames $\{\partial_\lambda\}$. Relativistic motion is located in the subbundle W_g of hyperboloids

$$g_{\mu\nu}(q)\dot{q}^\mu\dot{q}^\nu - 1 = 0 \tag{8}$$

of TQ. It is described by a second-order dynamic equation

$$\ddot{q}^\lambda = \Xi^\lambda(q^\mu, \dot{q}^\mu) \tag{9}$$

on Q, which preserves the subbundle (8), i.e.,

$$(\dot{q}^\lambda\partial_\lambda + \Xi^\lambda\dot{\partial}_\lambda)(g_{\mu\nu}\dot{q}^\mu\dot{q}^\nu - 1) = 0, \quad \dot{\partial}_\lambda = \partial/\partial\dot{q}^\lambda.$$

This condition holds if the right-hand side of the Eq. (9) takes the form

$$\Xi^\lambda = \left\{ {}_\mu{}^\lambda{}_\nu \right\} \dot{q}^\mu\dot{q}^\nu + F^\lambda,$$

where $\{{}_\mu{}^\lambda{}_\nu\}$ are Cristoffel symbols of a metric g, while F^λ obey the relation $g_{\mu\nu}F^\mu\dot{q}^\nu = 0$. In particular, if the dynamic equation (9) is a geodesic equation

$$\ddot{q}^\lambda = K^\lambda_\mu\dot{q}^\mu$$

with respect to a (nonlinear) connection

$$K = dq^\lambda \otimes \left(\partial_\lambda + K^\mu_\lambda\dot{\partial}_\mu\right)$$

on the tangent bundle $TQ \to Q$, this connections split into the sum

$$K^\lambda_\mu = \left\{ {}_\mu{}^\lambda{}_\nu \right\} \dot q^\nu + F^\lambda_\mu \tag{10}$$

of the Levi–Civita connection of g and a soldering form

$$F = g^{\lambda\nu} F_{\mu\nu}\, dq^\mu \otimes \dot\partial_\lambda, \quad F_{\mu\nu} = -F_{\nu\mu}.$$

As was mentioned above, the momentum phase space of RM on Q is the cotangent bundle T^*Q provided with the symplectic form Ω (1). Let H be a smooth real function on T^*Q such that the morphism

$$\widetilde{H} : T^*Q \to TQ, \quad \dot q^\mu = \partial^\mu H, \tag{11}$$

is a bundle isomorphism. Then the inverse image $N = \widetilde{H}^{-1}(W_g)$ of the subbundle of hyperboloids W_g (8) is a one-codimensional (consequently, coisotropic) closed imbedded subbundle of T^*Q given by the constraint $H_T = 0$ (2). We say that H is a relativistic Hamiltonian if the Poisson bracket $\{H, H_T\}$ vanishes on N. This means that the Hamiltonian vector field

$$\gamma = \partial^\lambda H \partial_\lambda - \partial_\lambda H \partial^\lambda \tag{12}$$

of H preserves the constraint N and, restricted to N, it obeys the Hamilton equation

$$\gamma \rfloor \Omega_N + i^*_N\, dH = 0 \tag{13}$$

of a Dirac constraint system on N with a Hamiltonian H.

The morphism (11) sends the vector field γ (12) onto the vector field

$$\gamma_T = \dot q^\lambda \partial_\lambda + (\partial^\mu H \partial^\lambda \partial_\mu H - \partial_\mu H \partial^\lambda \partial^\mu H)\dot\partial_\lambda$$

on TQ. This vector field defines the second-order dynamic equation

$$\ddot q^\lambda = \partial^\mu H \partial^\lambda \partial_\mu H - \partial_\mu H \partial^\lambda \partial^\mu H \tag{14}$$

on Q, which preserves the subbundle of hyperboloids (8).

Example 1. The following is a basic example of relativistic Hamiltonian systems. Put

$$H = \frac{1}{2m} g^{\mu\nu}(p_\mu - b_\mu)(p_\nu - b_\nu),$$

where m is a constant and $b_\mu\, dq^\mu$ is a covector field on Q. Then $H_T = 2m^{-1}H - 1$ and, hence, $\{H, H_T\} = 0$. The constraint $H_T = 0$ defines a closed imbedded one-codimensional subbundle N of T^*Q. The Hamilton equation (13) takes the form $\gamma \rfloor \Omega_N = 0$. Its solution (12) reads

$$\dot q^\alpha = \frac{1}{m} g^{\alpha\nu}(p_\nu - b_\nu),$$

$$\dot{p}_\alpha = -\frac{1}{2m}\partial_\alpha g^{\mu\nu}(p_\mu - b_\mu)(p_\nu - b_\nu) + \frac{1}{m}g^{\mu\nu}(p_\mu - b_\mu)\partial_\alpha b_\nu.$$

The corresponding second-order dynamic equation (14) on Q is

$$\ddot{q}^\lambda = \left\{ {}_\mu{}^\lambda{}_\nu \right\} \dot{q}^\mu \dot{q}^\nu - \frac{1}{m} g^{\lambda\nu} F_{\mu\nu}\dot{q}^\mu, \tag{15}$$

$$\left\{ {}_\mu{}^\lambda{}_\nu \right\} = -\frac{1}{2} g^{\lambda\beta}(\partial_\mu g_{\beta\nu} + \partial_\nu g_{\beta\mu} - \partial_\beta g_{\mu\nu}), \quad F_{\mu\nu} = \partial_\mu b_\nu - \partial_\nu b_\mu.$$

It is a geodesic equation with respect to the affine connection

$$K^\lambda_\mu = \left\{ {}_\mu{}^\lambda{}_\nu \right\} \dot{q}^\nu - \frac{1}{m} g^{\lambda\nu} F_{\mu\nu}$$

of type (10). For instance, let g be a metric gravitational field and let $b_\mu = eA_\mu$, where A_μ is an electromagnetic potential whose gauge holds fixed. Then the Eq. (15) is the well-known equation of motion of a relativistic massive charge in the presence of these fields.

Turn now to quantization of RM. We follow the standard geometric quantization of the cotangent bundle (Blattner, 1983; Śniatycki, 1980; Woodhouse, 1992). Because the canonical symplectic form Ω (1) on T^*Q is exact, the prequantum bundle is defined as a trivial complex line bundle C over T^*Q. Note that this bundle need no metaplectic correction since T^*X is endowed with canonical coordinates for the symplectic form Ω. Thus, C is a quantum bundle. Let its trivialization

$$C \cong T^*Q \times \mathbb{C} \tag{16}$$

hold fixed, and let $(q^\lambda, p_\lambda, c)$, $c \in \mathbb{C}$, be the associated bundle coordinates. Then one can treat sections of C (16) as smooth complex functions on T^*Q. Note that another trivialization of C leads to an equivalent quantization of T^*Q.

The Kostant–Souriau prequantization formula associates to each smooth real function $f \in C^\infty(T^*Q)$ on T^*Q the first-order differential operator

$$\hat{f} = -i\nabla_{\vartheta_f} + f \tag{17}$$

on sections of C, where $\vartheta_f = \partial^\lambda f \partial_\lambda - \partial_\lambda f \partial^\lambda$ is the Hamiltonian vector field of f and ∇ is the covariant differential with respect to a suitable $U(1)$-principal connection A on C. This connection preserves the Hermitian metric $g(c, c') = c\overline{c}'$ on C, and its curvature form obeys the prequantization condition $R = i\Omega$. For the sake of simplicity, let us assume that Q and, consequently, T^*Q is simply connected. Then the connection A up to gauge transformations is

$$A = dp_\lambda \otimes \partial^\lambda + dq^\lambda \otimes (\partial_\lambda + icp_\lambda \partial_c), \tag{18}$$

and the prequantization operators (17) read

$$\hat{f} = -i\vartheta_f + (f - p_\lambda \partial^\lambda f). \tag{19}$$

Let us choose the vertical polarization on T^*Q. It is the vertical tangent bundle VT^*Q of the fibration $\pi : T^*Q \to Q$. As was mentioned above, the corresponding quantum algebra $\mathcal{A} \subset C^\infty(T^*Q)$ consists of affine functions f (4) of momenta p_λ. Its representation by operators (19) is defined in the space E of sections ρ of the quantum bundle C of compact support, which obey the condition $\nabla_\vartheta \rho = 0$ for any vertical Hamiltonian vector field ϑ on T^*Q. This condition takes the form

$$\partial_\lambda f \partial^\lambda \rho = 0, \quad \forall f \in C^\infty(Q).$$

It follows that elements of E are independent of momenta and, consequently, fail to be compactly supported, unless $\rho = 0$. This is the well-known problem of Schrödinger quantization that is solved as follows (Blattner, 1983; Giachetta *et al.*, 2002a,b).

Let $i_Q : Q \to T^*Q$ be the canonical zero section of the cotangent bundle T^*Q. Let $C_Q = i_Q^* C$ be the pullback of the bundle C (16) over Q. It is a trivial complex line bundle $C_Q = Q \times \mathbb{C}$ provided with the pullback Hermitian metric $g(c, c') = c\overline{c}'$ and the pullback

$$A_Q = i_Q^* A = dq^\lambda \otimes (\partial_\lambda + icp_\lambda \partial_c)$$

of the connection A (18) on C. Sections of C_Q are smooth complex functions on Q, but this bundle need metaplectic correction.

Let the cohomology group $H^2(Q; \mathbb{Z}_2)$ of Q be trivial. Then a metalinear bundle $\mathcal{D}$ of complex half-forms on Q is defined. It admits the canonical lift of any vector field τ on Q such that the corresponding Lie derivative of its section reads

$$L_\tau = \tau^\lambda \partial_\lambda + \frac{1}{2} \partial_\lambda \tau^\lambda.$$

Let us consider the tensor product $Y = C_Q \otimes \mathcal{D}$ over Q. Since the Hamiltonian vector fields

$$\vartheta_f = a^\lambda \partial_\lambda - (p_\mu \partial_\lambda a^\mu + \partial_\lambda b)\partial^\lambda$$

of functions f (4) are projected onto Q, one can assign to each element f of the quantum algebra $\mathcal{A}$ the first-order differential operator

$$\hat{f} = \left(-i\overline{\nabla}_{\pi\vartheta_f} + f \right) \otimes \mathrm{Id} + \mathrm{Id} \otimes \mathrm{L}_{\pi\vartheta_f} = -ia^\lambda \partial_\lambda - \frac{i}{2} \partial_\lambda a^\lambda + b$$

on sections ρ_Q of Y. For the sake of simplicity, let us choose a trivial metalinear bundle $\mathcal{D} \to Q$ associated to the orientation of Q. Its sections can be written in the form $\rho Q = (-g)^{1/4} \psi$, where ψ are smooth complex functions on Q. Then the quantum algebra $\mathcal{A}$ can be represented by the operators $\hat{f}$ (5) in the space $\mathbb{C}^\infty(Q)$ of these functions. It is easily justified that these operators obey the Dirac condition

$$[\hat{f}, \hat{f}'] = -i\widehat{\{f, f'\}}.$$

Remark 1. One usually considers the subspace $E_Q \subset \mathbb{C}^\infty(Q)$ of functions of compact support. It is a pre-Hilbert space with respect to the nondegenerate Hermitian form

$$\langle\psi|\psi'\rangle = \int_Q \psi\overline{\psi'}(-g)^{1/2}\, d^{m+1}q.$$

It is readily observed that $\hat{f}$ (5) are symmetric operators $\hat{f} = \hat{f}^*$ in E_Q, i.e., $\langle \hat{f}\psi|\psi'\rangle = \langle\psi|\hat{f}\psi'\rangle$. In RM, the space E_Q however gets no physical meaning.

As was mentioned above, the function H_T (2) need not belong to the quantum algebra $\mathcal{A}$, but a polynomial function H_T can be quantized as an element of the enveloping algebra $\overline{\mathcal{A}}$ by operators $\widehat{H}_T$ (7). Then the quantum constraint (3) serves as a relativistic quantum equation.

Example 2. Let us consider a massive relativistic charge in Example 1 whose relativistic Hamiltonian is

$$H = \frac{1}{2m} g^{\mu\nu}(p_\mu - eA_\mu)(p_\nu - eA_\nu).$$

It defines the constraint

$$H_T = \frac{1}{m^2} g^{\mu\nu}(p_\mu - eA_\mu)(p_\nu - eA_\nu) - 1 = 0. \tag{20}$$

Let us represent the function H_T (20) as the symmetric product

$$\begin{aligned} H_T = {} & \frac{(-g)^{-1/4}}{m} \cdot (p_\mu - eA_\mu) \cdot (-g)^{-1/4} \cdot g^{\mu\nu} \cdot (-g)^{-1/4} \cdot \\ & \times (p_\nu - eA_\nu) \cdot \frac{(-g)^{-1/4}}{m} - 1 \end{aligned}$$

of affine functions of momenta. It is quantized by the rule (7), where

$$(-g)^{-1/4} \circ \hat{\partial}_\alpha \circ (-g)^{-1/4} = -i\,\partial_\alpha.$$

Then the well-known relativistic quantum equation

$$(-g)^{-1/2}\left[(\partial_\mu - ieA_\mu)g^{\mu\nu}(-g)^{1/2}(\partial_\nu - ieA_\nu) + m^2\right]\psi = 0$$

is reproduced up to the factor $(-g)^{-1/2}$.

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